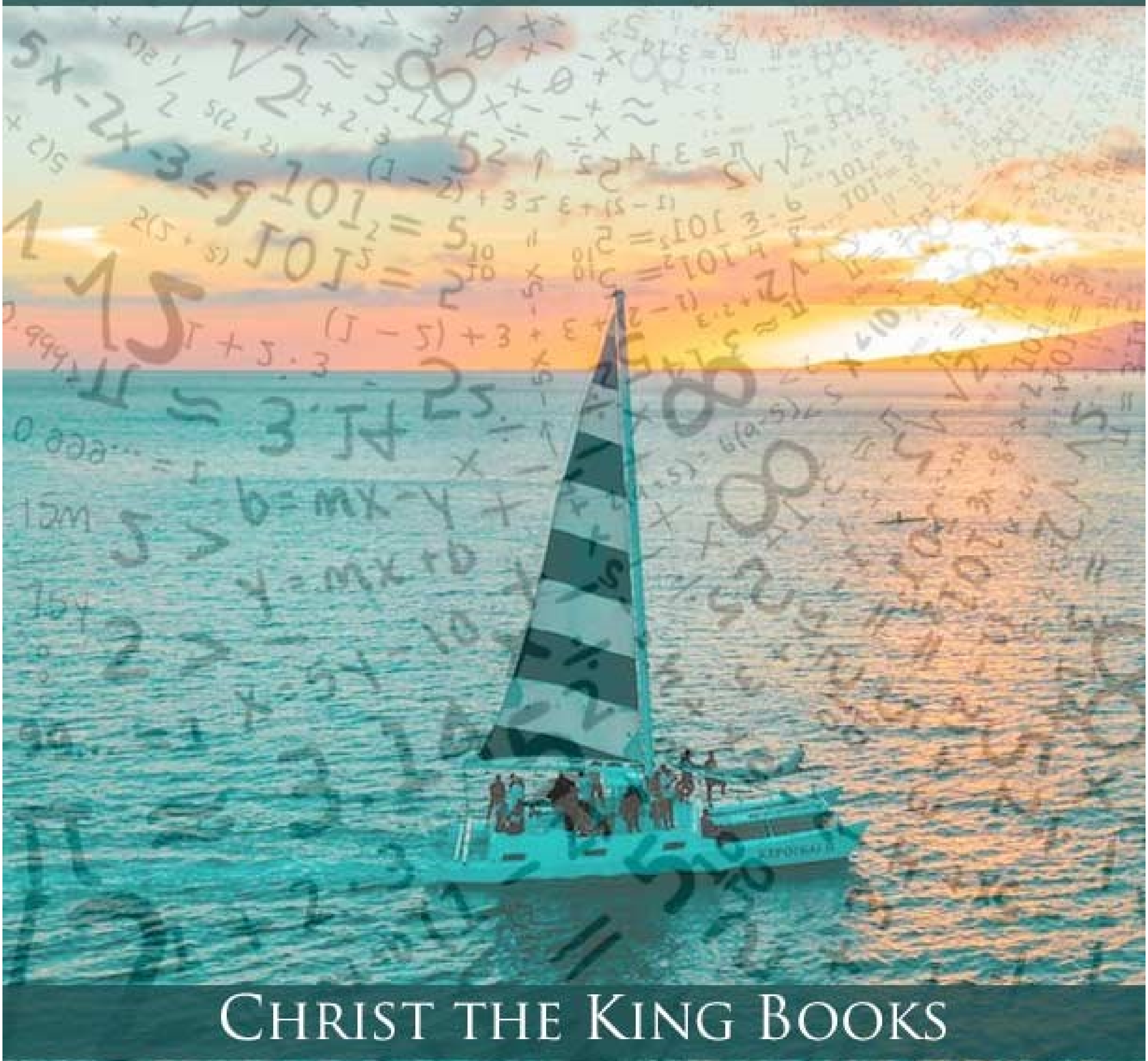
First Edition

PRE-ALGEBRA

A Catholic Approach



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Lesson 1 - Context for Algebra

What Is Algebra?

Welcome to the study of algebra! We hope you are excited to learn one of the most useful techniques in existence for coming to mathematical conclusions. Algebra is the key that opens the door to so many incredible discoveries in mathematics. Becoming comfortable with the algebraic processes allows you to see clearly what some of the greatest mathematical minds of all time discovered; because they relied heavily on its usefulness, you as their student will have to be able to use it in the same way in order to see the conclusions they arrived at by its help. Think of it as a new language, invented specially to communicate mathematical truths in a concise and helpful way. If you want to learn those truths, you first have to learn the language in which they are spoken.

But no need to worry, for learning algebra is in many ways much easier than learning a new language, and a big reason for this is there are *no exceptions to the rules*. This makes everything much more straightforward, and much easier to remember. Now, before we go any further, here is a definition of algebra in a few words:

Algebra **is a technique by which we come to the same conclusions that we would in mathematical demonstrations, but by using symbols and shortcuts instead of explaining every step.**

Preliminary Considerations

This definition may seem simple at first, but it needs some context. The mathematicians who invented this branch of mathematics did not do so at random; they had extensive knowledge of the other branches, and they used that knowledge to make something new. In order to understand "the conclusions of mathematical demonstrations," we need especially to look at two kinds of mathematics – geometry and arithmetic – and we also need to understand that there are two different kinds of quantity which they each study. Once we understand these concepts, we will see both how algebra builds on geometry and arithmetic, as well as how it differs from them. This will give us a clearer idea of what algebra itself is. First, let us look at the kinds of quantity.

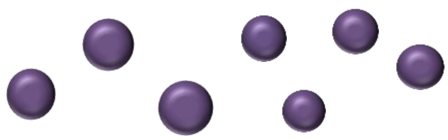
STOP! If you have not yet already read Christ the King Books' *Introduction to Modern Mathematics*, you should stop and do so now. (It can be purchased on the Christ the King Books website: www.ckbooks.org) **This short booklet is a quick read but it explains many critical concepts which you will need to understand in order to use this Pre-Algebra book.** Those concepts include:

What is a science? What is mathematics? What is the subject matter of mathematics (that is, what is mathematics about)? What are the ten categories of Aristotle, and why are they important? Which of them pertain to mathematics? What is a nature/essence? What are accidents? Which of these does mathematics concern itself with? What is quantity? What are the kinds of quantity which are the subject matter of mathematics? What is discreet quantity and what is continuous quantity? Why is this distinction important? What is the proper definition of *number*?

A Review of Quantity: Continuous or Discrete

As we mentioned, there are two kinds of quantity: the *discrete* and the *continuous*. **Discrete quantity** is what is made up of parts (or units) which do not share a common boundary, e.g., apples or numbers. These are discrete because they are not joined in any way; where one apple ends is not where another begins, and 5 itself does not share anything with 6 – they are completely independent. Another word for discrete quantity is **multitude**.

Continuous quantities, on the other hand, are not themselves made up of units. Examples of continuous quantities are lines and surfaces. We can divide a line into any parts we please, and the point at which each part ends is the exact same point at which the next begins. Think about it: when you place a pencil mark exactly in the middle of a line, is that mark in the first half or the second half? The truth is that the mark is not "in" either of the halves. Instead, it is shared by both halves – it is at one and the same time the endpoint of both halves of the line. This is what it means to share a boundary, and this is what makes them continuous quantities. These kinds of quantities are called **magnitudes**.



Discrete

Continuous

Arithmetic is the science concerned with discrete quantity, and **geometry** is the science concerned with continuous quantity. Essentially, arithmetic is about numbers, while geometry is about magnitudes. You might be used to thinking of geometry as the study of shape, and this is true, in a way. Remember, one of the examples of continuous quantity was surface, but triangles and circles are kinds of surfaces. So, geometry studies shapes, but *insofar* as they are magnitudes. We will look at some examples of geometry in the next lesson when we study Euclid's first proposition. We will also look at examples of arithmetic in later lessons.

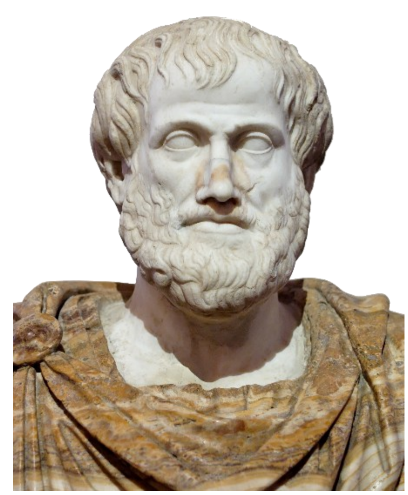
What does algebra concern itself with?

The techniques of algebra work on both kinds of quantity, as well as on useful things that are not really quantities at all, like zero. Thus, in this way, algebra can be viewed a branch of mathematics which unifies geometry and arithmetic; and in addition to unifying them, it does something special by expanding the sense of the word "number" to include zero and negatives – a very new idea in mathematics.

Now, the word "mathematics" can have multiple senses. Let us begin looking at what it meant before algebra entered the picture, because in order to see how algebra changed things, you must first see how things were before.

Math Before Algebra

Prior to the fall of the city of Constantinople to infidel Turks, mathematics was defined as the "science of quantity." Algebra is not mathematics in this sense, because algebra is not a science; rather it is a technique and investigation that is made possible *after* one has learned the various sciences of quantity "underneath algebra" such as arithmetic and geometry.



Aristotle

Alvaro Marques Hijazo, CC0, via Wikimedia Commons

Thus, before we dive into the study of algebra itself, we need to immerse ourselves in the ancient kind of mathematics for a little while to understand this brilliant new branch. To understand the ancient definition of mathematics, let us look briefly at the story of the characters involved in bringing it to us: Aristotle and Euclid.

Aristotle

After most of the glories of classical Greece – after the great victory over the Persians had been won, after the Athenian Democracy, after Socrates, and Plato – there lived a hard-working philosopher named Aristotle in the 300s B.C. He came to a deep understanding of the most important questions in life – questions ranging from how we know anything at all, to how we know

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Lesson 2 - Euclid's Principles

The Beginning of Euclid's Elements

Definitions

In Euclid's ***Elements***, he demonstrates truths about some magnitudes with which we already have some experience, such as circles and lines. However, in build up a careful science, he must first state exactly what they are, or what he means by them. Thus he begins his book with *Definitions*. **Definitions** are statements which state what something *is* (at least, what the author means by the term). Below are a few of the famous definitions with which Euclid begins.

> ***2***. *A line is breadthless length.*
>
> ***15***. *A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.*
>
> ***19.*** *Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.*
>
> ***20***. *Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.*

Good definitions explain what it is about the thing being defined that makes it have the name it does. So, what do we mean by "line"? A breadthless length. Why can you call this thing in front of you a line? *Because* it is a breadthless length.

Again, it is important to note that definitions merely say *what* something *is*. **They do not prove that something exists.** For example, Euclid could have defined a unicorn (a mythical horse with a horn on its head). This would not mean unicorns exist.

Most of the 13 Books in Euclid's *Elements* begin with definitions.

Postulates

Besides the definitions, Euclid also begins with statements called *Postulates*. **Postulates** are statements an author does not think he needs to prove, and actions he does not think he needs to prove he can do. The word *postulate* means a "demand", and Euclid uses this word since these are propositions Euclid thinks he just needs to ask the student to accept, or allow. Examples are:

> *To describe a circle with any center and radius.*
>
> *To draw a line from any point to any point.*
>
> *That all right angles equal one another.*

What do you think? Could these postulates could be proven, yet Euclid just skips doing so? If these statements could be proven, how could that be done? If not, why not?

If the reader will not accept the postulates, there is no way that Euclid can go on to establish science in that person's mind. For just as the definitions are critical for the proofs, so are the postulates.

Common Notions

In addition to Definitions and Postulates, Euclid also sets down five *Common Notions*. By **common notions**, Euclid means certain truths that are self-evident; they have no explanatory reasoning beyond understanding the meaning of the truth itself. All five of these apply to algebra, since algebra pertains to both magnitudes and numbers.

Before you read these common notions, however, it is important to note that **"equality" and "shape" are very different concepts.** "Shape" often involves the notion of quantity (for example, a triangle involves the number 3 but a rectangle the number 4), but shape also involves *quality* (for example, straight vs. curved, the way things are positioned relative to one another, etc). But "equality" is exclusively a matter of *quantity*; when mathematicians talk about things being "equal", they are speaking of the length, area, volume, etc.

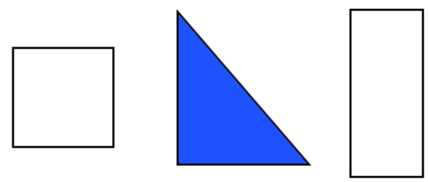
Common Notion #1: Mutual equality

"*Things which equal the same thing also equal one another*."

That square is the same size as that triangle.

That rectangle is also the same as that same triangle.

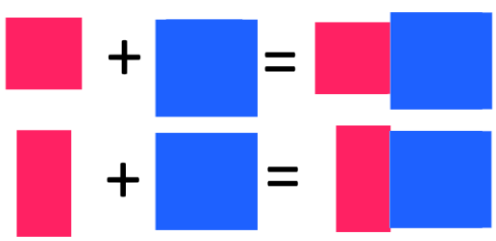
Therefore, that square must be the same size as that rectangle.



Common Notion #2: Adding equals to equals

Probably the most important Euclidean Common Notion for Algebra is this:

*"If **equals** be added to **equals**, the **wholes** are equal."*



Let it be that the **red** square and the **red** rectangle shown here are the same size (they are "***equals***").

If we add a blue square (a separate set of "***equals***") to each of the red shapes, the resulting wholes will be the same size.

Common Notion #3: Subtracting equals from equals

Euclid's Common Notion #3 is a simple variation on Common Notion #2:

*"If equals be **subtracted** from equals, the wholes are equal"*

Common Notion #4: Coincidence

"*Things which **coincide** with one another are equal to one another*."

Consider the two triangles below. If we take the one on the right and "slide it over" the one on the left, and the two figures then line up perfectly, with neither one "sticking out", this is the most straightforward and intuitive way to know things are equal in size. Of course, this only works if things are the same shape. Above we distinguished the concepts of equality and shape; two things coincide only when they are both equal and the same shape.

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Lesson 4 - Where Did Algebra Come From?

We just took a brief journey above into the country of mathematical demonstration – seeing true science step by step. We are now going to learn about that mysterious Arabic stranger called algebra.[5] With the help of algebra, we will turn a mathematical demonstration into something very simple. Algebra, in general, has two things: special *notation*, and certain *rules*. These things did not come out of nowhere: they developed out of the old way of doing mathematics.

But in the many centuries after Euclid, mathematicians in various nations went in "different directions" on how to simplify calculations and formulas. One of these peoples/nations were the Arabians living in different places.

The Arabs

Although there is evidence the Phoenicians and some others had been making progress in symbolic mathematics centuries before even Euclid, it was not until about 820 A.D. – over 1,000 years after Euclid's time, that an Arab Moslem[6] named Al-Kwarizmi living in the ancient city of Baghdad, wrote a book called *The Compendious Book on Calculation by Completion and Balancing*. In Arab, this title reads, "al-Kitāb al-Mukhtaṣar fī Ḥisāb **al-Jabr** wal-Muqābalah." Can you spot the word in that title from which we get the word "algebra"? The Arabic word *al-Jabr* means "completion" or "rejoining" and this gives us our modern English word "algebra."

Al-Kwarizmi's contribution to the development of algebra

Before Al-Kwarizmi's time, men had discovered some primitive ways to simplify geometry, and express geometrical truths in various shorthand notations. But Al-Kwarizmi turned algebra into an art all of its own, independent of geometry. In fact, for the Arabs, algebra became a thing much more tied to numbers than geometry. For his contributions to the development of the subject, many math historians call Al-Kwarizmi "the father of algebra."

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[5] Again, we trust the reader has already read Christ the King Books' short work called *Introduction to Modern Mathematics*, in which the reader is given an overview of mathematics, the various kinds of quantities, and the various branches of mathematics and their origins. www.ckbooks.org

[6] Although we are grateful to these Arabs that made such great progress in mathematics, we must never forget or lose our horror of their diabolical sect, always at war with the Catholic Church and Catholic Faith. Consider the hundreds of thousands, if not millions, of Catholics killed by Moslems in their wars of conquest in which they forced their victims to convert to Islam, or die. Think, for example, of the famous Battle of Lepanto in 1571 which saved Catholic Europe from an almost imminent takeover. Here are some quotes that will put us in the right state of mind. St. Francis of Assisi was on fire to bring the Moslems into the Catholic Church, outside of which there is no salvation. He boldly went to the Muslim Sultan Malek-el-Kamil and told him: "**We have come to preach faith in Jesus Christ to you, that you will renounce Mohammad, that wicked slave of the devil, and obtain everlasting life like us**." (*Saint Francis of Assisi*, by Omer Englebert, Franciscan Herald Press, 1966, Pg. 178-9) Pope Callixtus III called Islam "**the diabolical sect of the reprobate and faithless Mahomet (Mohammad)**." (*History of the Popes*, Ludwig) The Ecumenical Council of Basel (1431-49) called Islam "**the abominable sect of Mahomet.**" (*Council of Basel*, *Session 19*) St. Peter Mavimenus, being encouraged by some Arabs to convert to Islam, told them the following, for which they martyred him: "**Whoever does not embrace the Catholic religion will be damned, as was your false prophet Mohammed**." (*Roman Martyrology*, February 21st.) Many other such quotes could be given.

	
A statue of Al-Kwarizmi at the City of Madrid University	**The title page from Al-Kwarizmi's "The Compendious Book on Calculation by Completion and Balancing"**
Zarateman, CC0, via Wikimedia Commons	*Al-Khwarizmi, Public domain, via Wikimedia Commons*

The French

Some 800 years after Al-Kwarizmi, two French mathematicians – Francois Viete (1540 – 1603) and Rene Descartes (1596 – 1650), moved geometry much closer to modern algebra terminology and techniques.

Viete

Viete discovered that many problems that were wordy, complex, and cumbersome in traditional Euclidean-geometry style could be greatly simplified but using a different and simpler letter system. He knew about the Arabic work in algebra, but wanted to link algebra back to its geometric roots. Viete "found a way to adapt the numerical algebra of his time to a geometrical setting. By working abstractly with higher-dimensional magnitudes and by resolving proportions into equations, he laid the foundation for a new algebra. This new algebra, which he called *logistice speciosa*, was not just another step towards modern algebra. It was a complete overhaul of the very foundation of the art."[7] Viète was also the first mathematician to explore beyond the third dimension in geometry. In other words, before his time, geometers such as Euclid could only imagine in three dimensions, giving us lines, two-dimensional figures such as rectangles and so on, as well as solids such as spheres, prisms, and other three-dimensional figures. But Viete found a way with symbols to move beyond this, into four dimensions and beyond – "figures" which could be mathematically represented with symbols, but impossible to exist, and which proved useful for numeric calculations.

Viete used capital letters for his symbols – vowels A, E, I, O, U, and Y were used to represent unknown quantities, while the consonants were used to represent known quantities.

7 "François Viète's revolution in algebra" Professor Jeffrey Oaks from the University of Indianapolis https://researchoutreach.org/articles/francois-viete-modern-algebra/

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Substituting numbers for the lines

Remember that the (·) symbol here does not mean "times" (that is, multiplication), but means something about rectangles. The (^{2}) sign usually means something multiplied by itself, but in a geometric proposition, this too must mean something different, since we cannot multiply lines. Now let us see what happens when we substitute numbers for the lines, and use (·) to mean multiplication, and (^{2}) to mean what you get when you multiply the thing next to it by itself. Let us substitute the following numbers for the lines **5** for ***BC,*** **3** for ***BF,*** and **2** for ***FC***

Our paraphrase	Euclid's lines/figures converted to numbers instead
"The number 5 is equal to the number 3 plus the number 2."	$5 = 3 + 2$
"The number 5 multiplied by the number 2 is equal to the number 3 multiplied by the number 2 plus the square of the number 2."	$5 \cdot 2 = 3 \cdot 2 + 2^2$
"Subtract the square of the number 2 from each side of the above equation."	$-\ 2^2 = -\ 2^2$
"That leaves us with this: If we subtract square of the number 2 from the number 5 multiplied by the number 2 then the result is equal to the number 3 multiplied by the number 2."	$5 \cdot 2 - 2^2 = 3 \cdot 2$

Is the conclusion (the last line above) true? It certainly is:

$5 \cdot 2 - 2^2 = 3 \cdot 2$	
$5 \cdot 2 - 4 = 6$	
$10 - 4 = 6$	

That is the unexpected thing that happens when you use symbols cleverly: you can make it so the same writing is true, whether the symbols and letters refer to magnitudes or to numbers. You can do this even though the reasons for your conclusions are completely different. What we have here is like two entire languages, where if you reason correctly in one language, the words will also be true in the other one.

The one language is the symbol language of arithmetic, where (·) means multiplication and so forth, like we are used to. The other language is the symbol language of geometry tweaked a bit by the people who invented algebra, where (·) tells you to "make a rectangle with the two lines."

Why did this work? Why can numbers masquerade as lines if you use these special symbols? We have not yet seen the reason why this coincidence occurs.

Lesson Review

We have taken a very slow approach to algebra in this book, it is true. But that is because algebra is so very important in all the mathematics you will be doing from here on out; we thought it vital that you have a good, strong foundation for it. If you do not begin learning the rules of algebra while knowing exactly what it is you are doing, you will probably find it quite difficult to understand much of algebra at all: the rules will seem arbitrary and difficult to remember, the goal of the operations will be hard to see, and generally the experience will be less pleasant and interesting. Understanding what you are doing and what the point of algebra is before you begin can really make a world of difference; rather than being a chore, it could become your favorite subject! So, one last time, before we dive into doing some algebra, we will review what we have learned so far.

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Substitution Example #3: Parentheticals

A parenthetical just evaluates to the quantity that results from the operation(s) inside the parentheses. That is, parentheticals behave like single numbers or letters, because they represent just one quantity. Thus, we can also use substitution with parentheticals.

If this equation is true…	a	$=$	$x+y$
…and this equation is also true …	$m+a+p$	$=$	k
…then we can write this instead:	$m+(x+y)+p$	$=$	k

Keep the Parentheses

Every careful algebra teacher in history has recommended that beginners (and even advanced!) put explicit, visible parentheses around everything as they substitute that value in place of whatever it being replaced – even when substituting single letters and specified numbers like "5." **Firmly keeping this simple rule – even though it might seem to you like extra work – will prevent countless mistakes as you go forward with algebra.** In fact, there is a case in which it is mathematically necessary: when the term you are substituting is being subtracted, and you are replacing it with a parenthetical. In such a case, you *must* keep the parentheses. For example:

Correct – using parentheses when substituting

$$\boldsymbol{a-b=x}$$
$$\boldsymbol{b=c+d}$$
$$\boldsymbol{a-(c+d)=x}$$

Incorrect – no parentheses! This will definitely generate the wrong answer!

$$\boldsymbol{a-b=x}$$
$$\boldsymbol{b=c+d}$$
$$\boldsymbol{a-c+d=x}$$

Below are some equations in which you can see the difference between using parentheses and not doing so. Do not be concerned yet if you do understand *how* these calculations follow. You will learn the proper rules for subtraction soon. For now, simply concentrate that using and not using parentheses can give very different results!

$$\mathbf{7-2+3=8}$$
$$\mathbf{7-(2+3)=2}$$

$$\mathbf{10-4+6=12}$$
$$\mathbf{10-(4+6)=0}$$

Again, remember that when something is being subtracted, its replacement *must* be in parentheses. If you simply get into the habit of automatically putting parentheses around every value you substitute in, you will never have to worry about forgetting the rule!

A tricky restriction about substitution!

This also leads into another restriction. Just for right now, we **cannot substitute in this situation**:

Suppose	$b + c$	$=$	y
And further suppose that	$a - b + c$	$=$	x
We CANNOT substitute! This would be wrong!	$a - (y)$	$=$	x

We cannot substitute because $b + c$ is not in parentheses on the left in the second equation. On the left side of that second equation we see $a - b + c$. Those symbols taken together do **NOT** indicate to subtract the **sum** $b + c$; rather, those symbols tell us to first subtract b (only), and then **afterwards** to add c.

In a later lesson, we will learn some manipulations that allow us to change these equations so that we can substitute something; but for the moment we must just remember that substitution is not possible in this situation.

Quick Check – Test Your Understanding

Substitute. The answer is given upside-down, underneath.

Problem 1	Problem 2	Problem 3
$a + b + c = x$ $x + y = z$	$q - r - s = t$ $r - s = u$	$j - k + l = m$ $n = j - k + l - p$
$(a + b + c) + y = z$	*no substitution possible!*	$n = (m) - p$

More Examples

Again, substitution is probably the most common algebraic manipulation. One must practice, practice, practice to avoid mistakes which can cause a student to arrive at a wrong answer after much work in an algebra problem. Thus let us look at some more examples so you can become more comfortable with this process.

Example 1

1. $a + b + c = d, \quad b + c = n + p$

We see that there are terms in the left equation which are equal to something else in the right equation:

2. $a + b + c = d, \quad b + c = n + p$

So, we can just switch them out, remembering to put parentheses around them.

3. $a + (n + p) = d$

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Lesson 14 - Multiplication and Division of Numbers

In the last chapter, the lessons taught us much about adding and subtracting. When you work with these two operations, you do not need to consider whether the quantities are *discrete* or *continuous*.

Now, however, we are going to talk about multiplying and dividing, which you can only do with the primary kind of discrete quantity – i.e., numbers.

We will first look at the definitions of multiplying and dividing, their signs, and their fundamental theorems. Then, in the next part, we will go over to continuous quantity and look at different meanings their signs have for them.

Getting Rid of the Sign for Multiplication

You no doubt are familiar with this sign: $\times$

Up until now in your math journey, you have used this to indicate multiplication. However, in algebra we like to conserve space and use symbols only when absolutely necessary – because there are already so many symbols and letters floating around. Therefore, a new convention arose: signifying multiplication simply by putting the terms (in this case, the *factors* of a multiplication) right next to each other like this:

$$\mathbf{3 \times y = 3y}$$

$$\mathbf{b \times (y+2) = b(y+2)}$$

$$\mathbf{7 \times 3 = 7(3)}$$

Remember that multiplication is nothing other than *adding a number (called the multiplicand) to itself as many times as there are units in another number (called the multiplier)*. The number you get from multiplying the multiplicand by the multiplier is called the *product*.

Division is Measuring

The symbols you have probably used until now for division look like this: $\div$ or $/$

However, instead of these symbols, algebra uses this horizontal line notation:

$$\mathbf{3 \div y = \frac{3}{y}}$$

$$\mathbf{b \div (y+2) = \frac{b}{y+2}}$$

$$\mathbf{a \div b = \frac{a}{b}}$$

The result of division is called the *quotient*. Division ("taking a quotient") is nothing other than *finding out how many times a number (called the divisor) fits into another number (called the dividend), and that is why division is effectively the same thing as measuring*.

Division and Multiplication Cancel Each Other Out

Just as addition and subtraction are opposite operations, so are division and multiplication. Let us look at some number examples first, to see what we can conclude:

$$\mathbf{3(4) = 12} \qquad \mathbf{4(5) = 20} \qquad \mathbf{2(10) = 20}$$

$$\frac{\mathbf{12}}{\mathbf{3}} = \mathbf{4} \qquad \frac{\mathbf{20}}{\mathbf{4}} = \mathbf{5} \qquad \frac{\mathbf{20}}{\mathbf{2}} = \mathbf{10}$$

We see here that when we multiply a number by another number, and then divide the product by that same number, we end up right where we started. In words, "ten multiplied by two is twenty; and twenty divided by two is ten again."

This truth can be written universally in algebraic language like this:

$$\text{If } \boldsymbol{ab = m}, \quad \text{then } \boldsymbol{a = \frac{m}{b}}$$

Algebraic Variables and Constants

What are algebraic variables?

Variables in algebra are symbols, almost always letters, that represent different values. They help us understand and solve problems in which one quantity is related to another. They are called *variables* being they can represent any valid value that quantity can take on; the variable therefore, can "vary" – thus the name.

For example, when driving down the road, the speedometer of the vehicle registers a certain speed. Speed is nothing other than the relation of distance and time. To calculate the average speed of a vehicle, we simply ask, "How far did we travel in a given amount of time?" That can be written like this, with $\boldsymbol{s}$ representing the calculated speed, $\boldsymbol{d}$ the distance traveled, and $\boldsymbol{t}$ the amount of time.

$$\boldsymbol{s = \frac{d}{t}}$$

Since the distance we travel can vary, as well as the amount of time[12] we spend in the car, these are variable quantities; thus, we represent them with algebraic variables. Further, since speed is a concept which depends on distance and time, it too is a variable quantity.

Naming convention for algebraic variables

Following the customs of Rene Descartes, modern algebra generally reserves the lower-case letters in the second half of the alphabet (that is, n through z) to represent variable quantities (*variables*). So, technically, our example above of the formula for speed is non-customary. The letter d is not in the upper half of the alphabet but that letter represents distance, which can certain vary.

In the many manipulations we have already learned in previous lessons, we have used many different letters which we could have called variables. For simplicity, we have up to this point simply called them "letters" instead of using the proper term of *variables*.

[12] It turns out that the standard units of time used for speed is either the second or the hour, so speed can be written thus: $\boldsymbol{s} = \frac{\boldsymbol{d}}{\text{1 hour}}$ But often in scientific contexts in which much finer precision is needed, the unit of time is the second $\frac{\boldsymbol{d}}{\text{second}}$.

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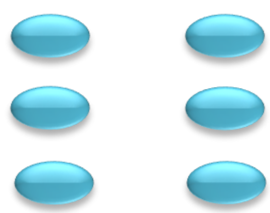
Lesson 17 - The Commutative Property: Multiplication

The Commutative Property of Multiplication

This is the *Commutative Property* of multiplication in words:

One expression multiplied by another is equal to the latter multiplied by the former.

What this means is that $ab = ba$, no matter what expressions a and b happen to be. Simply put, the order in multiplication does not matter. Hopefully this is intuitive; we are just stating it explicitly to make sure you understand it. Here is a visual to help you see how this is true:



$$(2)(3) = 6$$

$$(3)(2) = 6$$

Whether there are three groups of two, or two groups of three, the total remains the same.

We leave off the distinction between multiplier and multiplicand

The definition of multiplication says we should take the number of units in the *multiplier* (the number that tells you *how many times to multiply*), and then add the *multiplicand* (the quantity *being multiplied*) that many times. Thus, there is a logical distinction between the two terms. That logical distinction shows up in the way we arrange a diagram, as shown in the different diagrams above. But overall, the quantity is the same in the end even if you mix the two terms up.

However, in algebra, we ignore the logical distinction between multiplier and multiplicand. We simply call both of them *factors*.

Since it does not matter which factor comes first, you can distribute a sum as a factor any time in a multiplication – even if the sum comes first.

$$(\boldsymbol{b} + \boldsymbol{c})\boldsymbol{a}$$

$$\boldsymbol{ba} + \boldsymbol{ca}$$

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8. $\frac{2(x-4)}{2} = \frac{40}{2}$

9. $\frac{\cancel{2}(x-4)}{\cancel{2}} = \frac{\cancel{2}(20)}{\cancel{2}}$

10. $x - 4 = 20$

11. $x - 4 + 4 = 20 + 4$

12. $x - \cancel{4 + 4} = 20 + 4$

13. $\boldsymbol{x = 24}$

PEMDAS Order of Operations

When we are given an expression to **evaluate**, it is critical that we follow a certain order in the way we perform our variation mathematical operations / manipulations. This order is not something we ourselves choose, but rather, has long been agreed upon by mathematicians. In other words, mathematicians (including – and especially those who are performing Algebraic operations) write their expressions in a certain way, knowing that those expressions will be also be evaluated by other men in a certain, agreed upon way. This conventional order is called the **Order of Operations**, and it tells us which operations to complete first, then next, then next, and so on until we can simply / manipulate no further. It is a bit like the fact that English-speakers and writers long ago agreed that English text on the page should be read from left to right on the page, and from the top of the page to the bottom. (Other cultures have different conventions about this, such as reading from right to left, or even bottom to top.[19] So again, this concept of the Order of Operations is just simply a man-made agreement for the "language" of mathematics.) **It is extremely important that you memorize the accepted Order of Operations. Not all operations are commutative and associative, thus performing your operations in the wrong order can result in very incorrect or even absurd answers.**

The PEMDAS Acronym

You should read the following list in order, saying to yourself, "First I should handle the expressions contained within innermost parentheses first (if there are any); next, within any expression, I perform all multiplications (if there are any); next I work on the divisions…", and so on.

1) **P**arenthesis: We start with the innermost parentheses first.
2) **E**xponents: Do not be concerned about these yet; we will learn the concepts of exponents, powers, and bases next year in Algebra 1.[20]

---

[19] The Arabic and Hebrew languages, for example, are read from right to left. There are even a few languages in the Philippine Islands which are read from the bottom of the page to the top.

[20] Here is a nutshell introduction: An exponent indicates repeated multiplication. For example,

$$6^2 = 6 \cdot 6 = 36 \quad \text{and} \quad 4^3 = 4 \cdot 4 \cdot 4 = 64.$$

The exponent tells us how many factors of the base are being multiplied together.

3) **M**ultiplication / **D**ivision: Perform them in the order they appear in the expression from left to right.
4) **A**ddition / **S**ubtraction: Perform them in the order they appear in the expression from left to right.

Putting the first letters of each of the above operations together, we get the acronym PEMDAS ("Please Excuse My Daring Aunt Sally"). Memorize this acronym now to help you as you proceed further into advanced mathematics.

Example 6

Problem. Evaluate the following expression: $2 - 3 \cdot 5 + 4 \div 2 + (12 \div 2)$

Using the correct order of operations here is crucial. We consult our **PEMDAS** acronym, which instructs us to first look for parentheses, and we notice $(12 \div 2)$. Thus we evaluate that first:

Begin by evaluating the **p**arenthetical… $12 \div 2 = 6$ $\quad = 2 - 3 \cdot 5 + 4 \div 2 + \mathbf{6}$

Since we have not studied **e**xponents / powers yet, we do not worry about those. We skip to the next in the list: **m**ultiplication / **d**ivision.

Now evaluate the multiplication… $3 \cdot 5 = 15$ $\quad = 2 - \mathbf{15} + 4 \div 2 + 6$

Complete the division, next… $4 \div 2 = 2$ $\quad = 2 - 15 + \mathbf{2} + 6$

Now addition… $2 + 6 = 8$ and $-15 + 8 = -7$ $\quad = 2 - 15 + \mathbf{8}$

$= 2 - \mathbf{7}$

Finally, subtraction… $2 - 7 = -5$ $\quad = -5$

Example 7

Problem. Evaluate the following expression: $(3a + 4b \cdot 5 + 7) - 15b$

Begin by evaluating the **p**arenthetical $(3a + 4b \cdot 5 + 7)$. Within it, we see that there are additions and a multiplication. The multiplication comes first. After we do that, there is nothing more we can do within the parentheses, because the three terms are not "like" terms and thus cannot be joined.

$$(3a + 4b \cdot 5 + 7)$$
$$(3a + 20b + 7)$$

We reinsert our simplified parenthetical into the original problem.

$$(3a + 4b \cdot 5 + 7) - 15b$$
$$(3a + 20b + 7) - 15b$$

But since there is nothing but addition with the parentheses, we can drop them, and then join the terms which are "like" terms.

$$3a + 20b + 7 - 15b$$
$$3a + 5b + 7$$

Example 8: Nested parentheses

It is possible to have grouping symbols nested within grouping symbols. For example:

Problem. Evaluate the following expression: $7 + (52 - (3(17 - 12 \div 4) + 2 \cdot 5) \div 4)2 \div 2)$

With nested levels of grouping symbols, it can be hard to match up beginning and ending parentheses. To make it somewhat, we can rewrite so that every nested level alternates between parentheses and square brackets:

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Lesson 19 - The Four Operations on the Continuous

Moving From Discrete Quantity to Continuous

We have been working on inquiries proper to discrete quantity ever since we introduced multiplication and division. How, though, would you multiply continuous things – a line by a line, for example? How do you add a line to itself as many times as there are *units* in another *line*? Is there even a unit in a line? Although you can certainly multiply a magnitude by a *number* (e.g. "Make a new line which is three times as long as that line"), you cannot technically multiply or divide a magnitude by *another* magnitude. We raised some of these questions in a previous lesson, but now it is time to say more about these questions.

We have seen how, through algebraic manipulation, you can, with great ease and speed, reach conclusion after conclusion from an equation where the letters signify numbers. We are now going to leave numbers behind for a while, and see what it means to write equations about *magnitudes*, and draw conclusions about magnitudes using manipulation.

Adding and Subtracting Magnitudes

In the previous few lessons, all of the letters have stood for numbers. The proofs and problems began with "Let a be any number" or things like that. In this chapter, we are discussing algebra with magnitudes, thus, for now, all of the letters will stand for lines, surfaces, and solids.

However, there are conditions. For example, can you subtract this line from this circle?

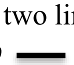
If you find a way of doing so, then you are better than a wizard. While it is true that these are both magnitudes, to add or subtract magnitudes, they must be of the same *kind* – there must be something common to both so that you can compare them. You cannot make a line greater than a cube, nor can you add or subtract a line and a cube.

For this reason, it will only make sense to add or subtract magnitudes of the same genus – line, surface, or solid.

Algebra With Straight Lines

Let us recall what it means to add and subtract straight lines. To add straight lines, you just put them end to end to make one longer straight line. To subtract, you put the shorter up against the longer, so they end in the same place, and your answer is the part that is left uncovered.

Suppose we have these two lines.

$\boldsymbol{a}$  $\boldsymbol{b}$ 

Then:

$\boldsymbol{a}+\boldsymbol{b}=$

$\boldsymbol{a}-\boldsymbol{b}=$

A New Meaning for Multiplication Notation

As we said before, you cannot multiply a line by a line. For, if you could, how many times would you add the line to itself? Since this makes no sense, we will not be using phrases like "a multiplied by b" where both a and b are magnitudes.

Consider for a moment how we are inclined to take words and give them new meanings – in a perfectly honest and decent way. We call the president the "head" of the country, but clearly, he is an entire man, and not just a head. We call a person healthy because he is alive and his body is working properly; but then we call a chicken salad healthy, even though we really hope it is not alive. We mean the salad is "healthy" in a different way – that is, it *promotes health in us*, rather than the salad itself being healthy.

We could go on and on thinking of different things with the same name, and often we will find that the second thing has the name because it is somehow like the first thing. Mankind has always done this in languages, and it has been extremely useful.

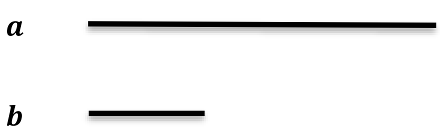
We are going to take multiplication notation and give it a new meaning that makes sense with magnitudes. We will see how this new meaning resembles the meaning that goes with numbers.

Multiplication of lines making an area

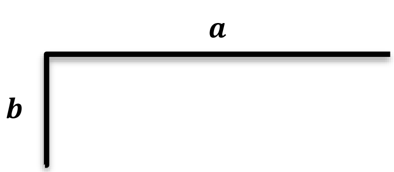
If a and b are lines, then when we see expressions like

we shall mean the product signifies the *rectangle contained* by a and b.

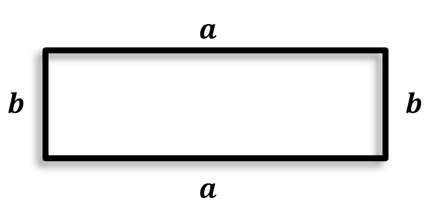
Making the rectangle contained by two straight lines is nothing other than picking up those straight lines,



Putting them at right angles,



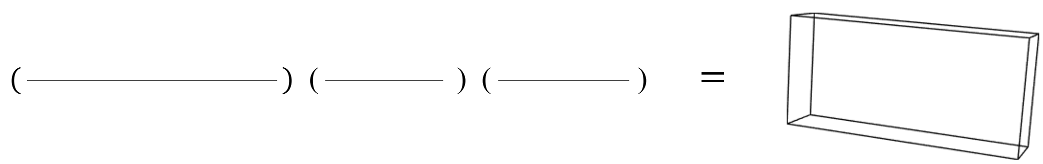
And drawing lines like this, from the bare end of each line, parallel to the other line:



Thus, we have enclosed an area. This is the rectangle contained by those lengths. Area is always written in squared units, such as "$200\ ft^2$ " or "$6\ m^2$". Hopefully it makes sense why: if we did not specify that the units are squared, i.e., areas, then we would end up with just length, like "$6m$". Six meters is a length, not an area.

Multiplication of lines making a volume

When we "multiply" three lines together, the first two line multiplied make an area (it is simplest to think of that area taking the shape of a rectangle; then multiplying that rectangle by the remaining line, yields a box:



Abstraction

This is a good place to remind you that in algebra (as in pure mathematics[21] in general) our concern is generally only with size. For example, notice that above, the rectangle *a* and the line *c* had not only length, but also had the attributes of a certain location, position, or orientation. However, we are not concerned with what we know about things with respect to place or position, but only the "*how much"* of those magnitudes. This ignoring of the specific circumstances is called "abstraction."

How to "Divide" a Rectangle by a Line

A rectangle has four sides, but the top and bottom equal each other, and the left side and right side equal each other. Therefore, there are in reality only two different lengths involved. Henceforth, we will refer to the two lengths as the *two* sides of a rectangle, even though there are four sides altogether in the normal way of speaking.

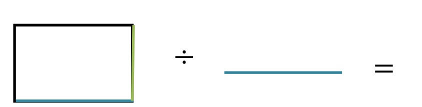
Here we have a rectangle, and its two sides have been drawn near it.

Since there is a special meaning of multiplication notation that lets you use it on magnitudes, there is also a special meaning of division sign that undoes what the multiplication does.

So, given a rectangle, we use division notation to ask "If *this* is the area of a rectangle, and *that* is one of its sides, how long is the other side?"

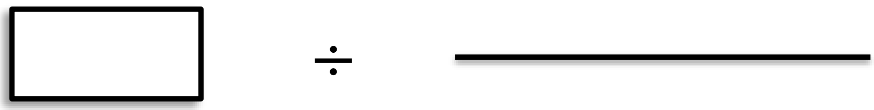
$$\frac{\text{rectangle}}{\text{one side}} = \text{the other side}$$

We keep in mind with this notation that, again, we cannot *truly* divide an area (like a rectangle) but a length (a line). Again, it is only convention to use these phrases like "dividing a rectangle by a line."



Sometimes you are given a rectangle, and a line that does not match either of its sides.

[21] The term "pure" means mathematics completely separated from any physical or material concepts. This term is opposed to the terms "mixed" or "applied" mathematical sciences. There are many "mixed" sciences, such as modern physics - which mixes pure mathematical concepts with concepts beyond pure mathematics such as movement, speed, weight, time, gravity, mass, etc. Astronomy is another example of an "applied" science.



In this situation, you cannot simply ask what the "*other*" side of the rectangle is, since the given line does not match either side of the rectangle. However, if the rectangle is reshaped so that after reshaping, the bottom side of the rectangle is equal to the line by which you want to divide, but the rectangle still has just as much area, then at that point we would be able to "divide" the rectangle by the line, as usual. Here is the above rectangle resized to match the line we had.

Because you can reshape any rectangle like this, you can therefore "divide" any rectangle by any line. Even if the line we want to "divide" by were one million miles long, that is fine! The resulting rectangle would just be of a height so tiny that it would be invisible to our eye – but it would have *some* height.

Dividing a Magnitude by a Magnitude of the Same Kind

How are we to interpret this?

$$\frac{\mathbf{1\ yard}}{\mathbf{1\ foot}}$$

By this we mean *"How many times does a foot fit into a yard?"* Instead of dividing, here we are *measuring*. That is, instead of starting with the whole yard, we focus first on the length called *foot;* and then, one by one, we place that length against the yard length, and see how many fit into it. Since both yard and foot are one-dimensional units (they are just length width no width), here we are measuring a magnitude by a magnitude.

We, of course, know that a foot fits into a yard three times, so a "yard ÷ a foot" is three – that is, a number.

Similarly, if you ask how many times a normal-size piece of paper fits on the top of the desk, or how many times this cupful can go into that bucket, you get a number. **A magnitude divided by a magnitude of the same kind yields a number**.

How Is Division the Reverse Operation of Multiplication with Magnitudes?

Because you can "divide" a rectangle by a line, you can now answer the backwards version of a multiplication problem: "What number do you multiply 3 by to get 12?" The answer is, of course, 4. Likewise we can ask, "What line do you 'multiply' line b by, to get rectangle r?"

$$? \cdot b = r$$

Think about when you multiply: The lines you multiply together to make a rectangle become its two sides. Therefore this question is asking for *the other side* of a rectangle with area r, when one of the sides is b. You can rewrite that same question, using the division sign as follows:

$$\frac{r}{b} = ?$$

So we see that the division sign with magnitudes, just as with numbers, does the reverse of multiplication.

Examples

Example 1

The paper has one side that is 8 inches long, while the other is 11 inches long. What is the total area of the paper?

Let us start with algebraically writing what we already know is true, and then writing in the values that we know. An area is composed of one side "multiplied" by the other side.

1. $Area = (side)(other\ side)$

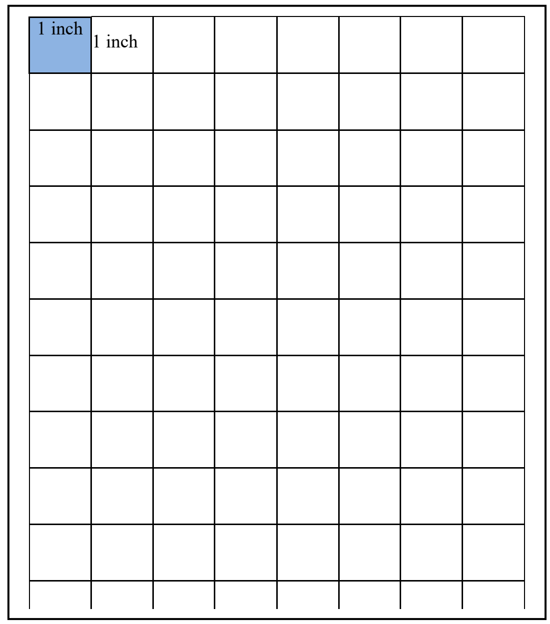
We are given the lengths of the sides already, so our equation looks like this:

2. $Area = (8\ inches)(11\ inches)$

And now all we need to do is multiply the numbers out to get our area.

3. $\mathbf{Area = 88}$ **inches x inches =**

$\mathbf{88\ square\ inches = 88\ in^2}$



There are some important things to note about that answer:

1) Notice that **we always include the units in an answer, if we know the unit**. Imagine how frustrating it would be if you asked someone, "How far is it to your friend's house?" and you were simply told, "7". Would you not ask, "7 WHAT? 7 miles? 7 blocks?" Later you ask a different person, "How far away is the sun from the earth?" She answers, "93 million." Somewhat frustrated now, you respond, "93 million WHAT?" "Oh, sorry, 93 million *miles*," she says. You are beginning to think people around you are a little crazy. But, hold on, because little do you know there is more to come!
2) Now that we know the importance of always specifying a unit, we take note of the answer to the area of the paper: "88 ***square inches***." This brings up an important point: **When we specify a unit, we must always use a unit the other person will understand**! Let us suppose the same day you ask your friend about his sportscar: "Hey Joseph, what is the top speed of your car?" He braggingly answers, "This car? It's *really* fast! It can do 150 slovajoks!" Thinking the world has gone mad, you answer, "What? What do you mean? What is a *slovajok*?" After now having three bad experiences in the same day, you run away saying a Hail Mary lest you explode in frustration. Joseph should have responded using a unit you understand – in fact, with a unit most men would be familiar with: "miles per hour" or "meters per hour" would have been nice. What do we learn from these three frustrating experiences? That **when answering questions about quantities, we should always include the unit, and a unit that the person will understand**, otherwise we are not really communicating our thoughts.
3) So what exactly is the unit of a "square inch"? The answer is very simple: it is a square the sides of which have a length of 1 inch. You can see the blue square inch unit in the figure above. (We are not claiming it is accurate to size; we are just looking for the idea here). So when we say the paper has an area of

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Lesson 28 - Dividing with Fractions

Dividing a Fraction by a Whole Number

In algebra, there are many times when you will need to divide something that is already being divided – that is, you will need to divide a fraction by another quantity. What you end up doing is dividing a fraction by another number. The rule for this is:

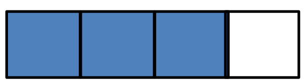
Dividing a fraction by a whole number is the same as multiplying the denominator of the fraction by the whole number.

Example 1

Divide $\frac{3}{4}$ by 2.

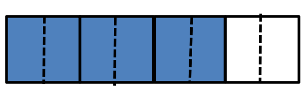
First, try to visualize the dividend (the thing to be divided)

We can visualize the problem as follows. Suppose we have four squares – each of them representing a unit called "a fourth", which in symbols, as written as $\frac{1}{4}$. We give it the name of "fourth" since **4** of them make a whole set. If we have three of those "fourth" (that is, $\frac{1}{4}$) unit squares filled, the total would be "three of our one-fourth units", or what we call $\frac{3}{4}$ for shorthand. It would look like the following figure. Think of a cardboard box with four compartments, three of which are filled:

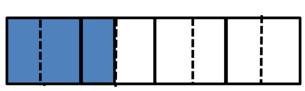


Now, what does dividing a fraction by 2 mean?

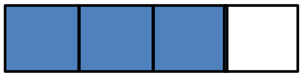
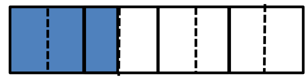
Now: what does it even mean to ask the question, "What is $\frac{3}{4}$ divided by 2" ? It would mean to somehow split the above three blue shaded areas into two equal parts. But there is an apparent problem! How can we get two parts (an even number), when there are currently three divisions (an odd number)? *Clearly, we will need to first subdivide the squares (including the three blue ones) into an even number of sub-parts; then we can evenly divide those resulting sub-parts into two groups*. Let us do that now:



Instead of 4 squares, we now have 8 rectangles – an *even* number of units. Because we now have an even number, we can divide those 6 rectangles into two sets of 3. But notice – *our unit has changed*. We are no longer counting squares! Now we are counting rectangles, called "eighths." Why do we choose that name? Because eight of them fill the whole cardboard box. So, half of the set of six would be called "three eighths", which written symbolically is $\frac{3}{8}$ and it looks like this:



Summing it up visually:

This  divided by 2, gives us this: 

What does the same problem look like in notation?

Before you can understand the notation below, keep this in mind: we could not evenly divide the three squares because three is an odd number. So we effectively *multiplied the number of units by two*, by simply by dividing each of the squares. Even though it seems contradictory to get a multiplication out of division, *the answer to the riddle is that we have smaller units afterwards. We divide bigger things* (e.g., squares)*, and by doing so, we multiply into smaller units* (e.g., half-squares). Here is what that looks like in notation. Notice that the *division* by two $\div\mathbf{2}$ becomes a *multiplication* in the denominator $\mathbf{4(2)}$

$$\frac{\mathbf{3}}{\mathbf{4}} \div \mathbf{2} = \frac{\mathbf{3}}{\mathbf{4(2)}} = \frac{\mathbf{3}}{\mathbf{8}}$$

Reciprocals

An extremely useful concept in mathematical manipulations is that of the **reciprocal.** The **reciprocal of any given number** (let us call the given number a) is some other number (let us call it b) such that $a\ times\ b = 1$. That definition can be a bit difficult to understand without an example. Ask yourself: What number, when multiplied by 4, will give a product of 1? $4\ times\ ? = 1$. There is only one answer: $\frac{1}{4}$. $4\ times\ \frac{1}{4} = 1$.

How to quickly find the reciprocal of any number

The reciprocal of a number is formed by taking that number and switching its numerator and denominator. For example, the reciprocal of 2 is $\frac{1}{2}$. This can be seen by first remembering that $2 = \frac{\mathbf{2}}{\mathbf{1}}$ (since every whole number can be expressed as a fraction with a denominator of 1). We can then switch the numerator and denominator to get the reciprocal of 2, which is $\frac{\mathbf{1}}{\mathbf{2}}$.

Here are some more sets of a number with its reciprocal. Remember, a whole number is the same thing as that number over 1.

$$\mathbf{3}\quad \frac{1}{3} \qquad \frac{\mathbf{2}}{\mathbf{3}}\quad \frac{3}{2} \qquad \mathbf{-27}\quad -\frac{1}{27} \qquad \boldsymbol{x}\quad \frac{1}{x} \qquad \frac{\mathbf{3y}}{\mathbf{7}}\quad \frac{7}{3y} \qquad \boldsymbol{a|14df|}\quad \frac{1}{a|14df|}$$

Using the concept of a reciprocal to understand "dividing a fraction by a number"

Another way to think of dividing a fraction by a number is this: When dividing a fraction by a number, we are effectively just multiplying the fraction by the *reciprocal* of the whole number.

Suppose we want to divide some fraction by the number 2. We can instead multiply the fraction by the reciprocal of 2, which is $\frac{1}{2}$.

$$\frac{\mathbf{12}}{\mathbf{3}} \div \mathbf{2} = \frac{\mathbf{12}}{\mathbf{3}}\left(\frac{1}{2}\right) = \frac{\mathbf{12(1)}}{3(2)} = \frac{\mathbf{12}}{\mathbf{6}} = 2$$

Keep reciprocals in mind as we move forward in this lesson, so that you can see the pattern in all three circumstances we talk about below.

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Chapter 5 - Solving Linear Equations

Lesson 30 - Revisiting Simplifying

How to Simplify

We will now learn a technique that has to do with *how things look on paper*, rather than *what they mean*. This technique is called *simplifying* an expression.

The purpose of simplifying an expression is to write the same quantity as briefly as possible with no parentheses, and without repletion of terms that end up cancelling out. We do this by:

1. removing parentheses,
2. combining like terms, such as numbers and multiples of the same variable
3. cancelling.

"But have we not already learned simplifying?"

These are all things we have already been doing when solving equations. However, it is important to really focus on doing this as much as possible as we move forward, especially as we tackle word problems. Keeping our equations as simple as possible will be a great help in not becoming confused.

You may have already noticed, but in algebra it seems we simplify *more* than we manipulate; so many times – especially with word problems – we don't actually need to manipulate an equation to find our answer. The answer is already there, we just need to simplify so we can see it clearer.

So, for these reasons, even though we have already been doing this for a while without focusing on it, we are going to drill into this technique further.

An example that shows how much simplifying can help

Here is an example in which simplifying makes a big difference in how easy it is to see the answer. We start with this equation, and we want to solve for m.

$$(3+4)(m+2n)-14n=1$$

One way to solve would be to immediately begin manipulating to isolate m, dealing with one term at a time. Act as if though you did not notice that you could simply add 3 and 4; let us then see what happens. Our lack of attention would have us do the following steps, which would end up looking like this:

$$(3+4)(m+2n)=1+14n$$

$$m+2n=\frac{1+14n}{3+4}$$

$$m=\frac{1+14n}{3+4}-2n$$

Technically, we are done, because we have isolated m. However, that is quite a complicated answer! Imagine if we had to substitute that value of$m$ into *another* equation. The result would not be very pretty, and rather tricky to solve. *There must be a better way!* There is, of course.

Simplify as You Go

Now let's try solving the same equation, but with simplifying as our priority. We will manipulate to isolate m, but first we will focus on simplifying everything that we can. We will be using the list we laid out above:

1. removing parentheses,
2. combining like terms, such as numbers and multiples of the same variable
3. cancelling.

$$(3+4)(m+2n)-14n=1$$

$$(7)(m+2n)-14n=1$$

$$(7m+14n)-14n=1$$

$$7m+\cancel{14n-14n}=1$$

$$7m=1$$

Up to this point, all we have done is simplify; no manipulations (substitutions, "doing something to both sides of the equation", etc.) have been performed, because nothing has been done to that solitary numeral 1 on the other side of the equation. Now that everything is all nice and simplified, we will perform a manipulation: we will divide both sides in order to isolate m:

$$\frac{7m}{7}=\frac{1}{7}$$

$$m=\frac{1}{7}$$

Now *there* is a simple answer! Although it may not be obvious to you, this simple answer is equal to the complicated one we got before, but this looks much nicer and easier to work with. If we had an equation that had the variable m in it, we could substitute this answer easily!

"I do not believe you. Those answers cannot be the same!"

Just in case you are a "doubting Thomas",[30] we will prove the above answers are equivalent, using algebraic manipulation. First, as shown below, we set them into an equation. The little question mark over the equal signs below indicates our "doubt." Now, if we can manipulate them until both sides of the equation are identical, we will have shown that the expressions are equal:

$$\frac{1+14n}{3+4} - 2n \stackrel{?}{=} \frac{1}{7}$$

$$\frac{1+14n}{3+4} - 2n \stackrel{?}{=} \frac{1}{7}$$

$$\frac{1+14n}{7} - 2n \stackrel{?}{=} \frac{1}{7}$$

$$7\left(\frac{1+14n}{7} - 2n\right) \stackrel{?}{=} 7\left(\frac{1}{7}\right)$$

$$1 + 14n - 14n \stackrel{?}{=} 1$$

$$1 + \cancel{14n - 14n} \stackrel{?}{=} 1$$

$$1 = 1$$

Now we know the two answers were in fact equal. But clearly, they do not look the same; and we much prefer the simpler looking one. It makes everything so much easier when you can work with less complicated expressions.

Again: Simplify as You Go

After every step in your work, it is useful to take every opportunity you can to remove parentheses, combine like terms, and cancel. This will speed up your work and make it easier, as well as do much to eliminate the opportunities for errors in manipulation.

Keep in mind that, sometimes, in order to simplify something in one way, you cannot simplify it another way. For example, in order to cancel, you need to see the different factors of an expression; but simplifying can also include multiplying out the factors in order to see if something cancels that way.

[30] A reference to the Apostle St. Thomas who doubted Our Lord had truly risen from the dead, until it was proven to him by Our Lord showing him, and having him insert his finger into His Sacred Wounds.

$$\frac{5(2x+4)}{5(x+2)} = y \qquad \frac{5(2)(x+2)}{5(x+2)} = y \qquad 2 = y$$

Here, it would have taken several more steps to simplify if we had decided to distribute (that is, multiply out the 5 in the numerator and denominator). A few good tips to keep in mind are:

When you are dealing with division, it is best to first focus on factoring to see if you can get a match in the numerator and denominator; then you can cancel like terms.

In general, if the operation you are about to perform will create more terms, and those new terms do not cancel out, it is probably best to leave things alone.

So, overall, just try to notice when what you are about to do will be helpful or not. But it is quite alright if you simplify and then discover it would have been faster if you hadn't; you can generally recover from this, and you will just remember that inconvenience for next time.

Examples

Example 1

Simplify.

$$\frac{5+2x+3+3x+8+3x}{2(x+2)}$$

We are going to begin by combining like terms and see where we can go from there.

1. $\frac{5+2x+3+3x+8+3x}{2(x+2)}$
2. $\frac{2x+3x+3x+3+5+8}{2(x+2)}$
3. $\frac{8x+16}{2(x+2)}$

Now it looks like we can factor the top to get some cancellation.

4. $\frac{(2)(4)(x+2)}{2(x+2)}$
5. $\frac{\cancel{(2)}(4)\cancel{(x+2)}}{\cancel{2}\cancel{(x+2)}}$
6. $\frac{4}{1}$
7. $\mathbf{4}$

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Equation for *any* triangle (variables)	$a = \frac{bh}{2}$
Equation for the triangle on the wall (the unknowns):	$a_1 = \frac{b_1 h_1}{2}$
Now we can insert the given lengths for the knowns. **Notice**: Always insert the units, too! *Not* doing so is a fertile source of errors.	$a_1 = \frac{3ft \cdot 6ft}{2}$
Thence we need only solve for the remaining unknown and combine and simplify as we go.	
Do the multiplication.	$a_1 = \frac{18\, square\, feet}{2}$
Simplify the fraction.	$a_1 = 9\, square\, feet$

But a_1 was the area of our triangle on the wall, so now we know the exact value of its area.

Why do we bother with subscripts?

The essential thing to keep in mind here is that when we particularize an equation with subscripted variables on a particular setup, strictly speaking, we are transforming it into an equation with *knowns* and *unknowns*. For example, concerning our spray-paint triangle, if we had done this (where b is the base of *any* right triangle)

$$b = 3\, feet$$

instead of this,

$$b_1 = 3\, feet$$

then we would have an absurdity, because it is NOT true that the base of ***any and every*** triangle is 3 $feet$. Rather, we are talking about a specific line – the line b_1 – ones of the sides of the particular triangle on the wall.

"Plugging in" Directly for the Variables

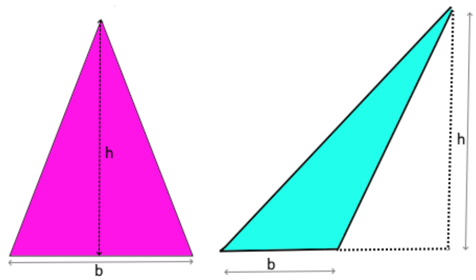
When you take an equation and substitute particular values for the variables (as we just did in the example above), we say that you "**plugged in** the values." An equation with variables into which we often plug in values is called a **formula**.[34] It is called a formula because it provides the form for many equations about particulars, just as

$$a^2 + b^2 = c^2 \quad \text{does for} \quad {a_1}^2 + {b_1}^2 = {c_1}^2.$$

Additionally, we need to remember this if we use the same formula more than once in the same problem. We will need to use subscripts or some other mark to distinguish between the corresponding unknowns. We could also use completely different letters.

[34] The plural is form is either *formulas* or *formulae* depending on whether one chooses to fit in with the crowd, or to instead sound correct, Latin-ish, and fancy.

Solutions to Equations



The formula for the area of a triangle is this:

$$area = \frac{1}{2} base \cdot height$$

Now there are many different triangles out there, and this formula holds for any and all of them. If we were to make a list of all the triangles in existence, we could write the values for their base, height, and areas. We would list these quantities in the same order every time, and put parentheses around them, where b is the base, h the height, a the area:

$$(b, h, a)$$

Now not just "any old" three numbers will work as members of this list. We can use the equation to figure out which ones work and which ones do not. Suppose we have these in a list:

$$(2, 2, 2)$$
$$(3, 3, 4)$$
$$(5, 56, 9)$$
$$(4, 3, 6)$$

If the formula does not work on some sets of values, that set cannot possibly refer to any triangle. Let us check. The sets of values in red are "lemons" – they do not work!

(b, h, a)	$\frac{bh}{2} = a$	
$(2, 2, 2)$	$\frac{2 \cdot 2}{2} = 2$	☑
$(3, 3, 4)$	$\frac{3 \cdot 3}{2} = 4$	☒
$(5, 56, 9)$	$\frac{5 \cdot 56}{2} = 9$	☒
$(4, 3, 6)$	$\frac{4 \cdot 3}{2} = 6$	☑

The scope of the values of algebraic variables in formulas and equations

The main point we are making here is this: when one sees a formula such as

$$a = \frac{bh}{2}$$

it is true that there is an infinite number of sets of values that will satisfy the formula. *In that sense*, the letters in the formula can stand for *any* number. But there is also an even "bigger infinite list" (try to wrap your mind around that!) of sets of three values that will *not* satisfy the formula. For example, looking at the above list of "lemons" in red, if we choose 5 for the base and 56 for the height, we are NOT free to choose 9 for the area – those three values will not work in combination with one another.

This is the key point: although each of the variables in the equation can stand for any number, yet that does not mean the *other* variables in that equation can also stand for *any* number, *after we have chosen a particular value for the other variable*. Picking a particular value for one variable in an equation generally "locks down" the possibilities for the valid values of all the other variables. That is, after all, what makes the equation

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Lesson 37 - Solving Word Problems

What is a word problem?

It is not the usual way of things for mathematicians and scientists to suddenly get a telegram with beautiful and true equations describing the reality God created. Instead, they communicate with one another through words, and in the course of much verbiage and explanations, one sees mathematical expressions interspersed.

We students of algebra should expect nothing else than this. Reality is complex, and we have to first describe it with words, try to discern general patterns, which we can then put into algebraic notation. For example, we might say something like, "Bob, can you help me with this problem, please? I am trying to figure out how many miles I can travel, on average, per gallon of fuel I pump into my trusty, rusty old pickup truck." "Oh, sure Nathan. Well, next time you go to the gas station, first check the odometer… Then do such-and-such…"

So again, we first have to translate sentences (with many words) into equations as simple as possible algebraic formula. Then we need to solve that formula for whatever value we are interested in.

Problem. If a man walks $\boldsymbol{y}$ miles per hour, how many miles will he walk in $\boldsymbol{x}$ hours?

We think: "In any given hours, the man travels y miles, on average. Thus for each hour that passes, I will add y miles to the total. But I am also told that he walks for x hours. I can sense that multiplication is needed here. Yes – I've got it! The total number of miles can be expressed as the multiplication of y by x. In algebra, this can be written simply as xy. The answer is that the man will walk xy miles in x hours of time."

Hints for solving word problems

In several of the remaining lessons in this book, we will be revisiting word problems and providing sets of hints and techniques for solving them. Here is the first of those sets.

1. Always keep the goal in mind.

A very typical problem for students when reading word problems is to become so frightened and intimidated by all the verbiage, numbers, and terms, that the student freezes and gives up. To avoid this, try as a first step to ignore all those complications and to focus on the goal. What is this problem asking you to do? What is the information sought? What would *general form* would the answer take? Write that down so your mind has something to move toward.

2. When analyzing the problem, try to abstract from everything not applicable

Try to ignore everything in the problem which is not directly applicable so that we can "see the forest through the trees." One thing that can often be ignored is the very nature of what is being calculated. For example, the above problem about walking would be just the same if we changed miles and hours into something else: "If there are $\boldsymbol{y}$ parties every day, how many parties will there be in $\boldsymbol{x}$ days? There will be yx parties in x days."

Some more examples of word problems

Problem. In $\boldsymbol{x}$ years a man will be 40 years old; what is his present age?

We first identify what is being asked, and what the form of the answer would be. We are seeking his current age, and the answer should look something like, "$present\ age = 20\ years\ old$" or something like that. As for ignoring things, there is nothing here to set aside.

So we begin. First of all, his present age must be less than 40, since he will be 40 after some number of years. Thus we are looking for an answer smaller than 40. His age after x years will be (his age now + x). Therefore, his age now must be $(40 - x)$.

Problem. A regiment of men is drawn up in $\boldsymbol{r}$ ranks of 80 men each, and there are 15 men leftover. How many men are there in the regiment?

There is nothing to ignore here, and the answer should take the general form "There are x number of men in the regiment."

We begin. A regiment is a group of men in the military. This regiment is apparently made up of subdivisions called *ranks*. Here, evidently, a *rank* means 80 men and the problem states there are r number of ranks. This means there will be r times 80 men. This can be written as $(80)\mathrm{r}$, or more simply, $80\mathrm{r}$. But there will also be 15 men left over. So, altogether there are $80\mathrm{r} + 15$ men in the regiment.

Problem. Each emir[35] brought five luxury cars to the party. If there are 35 luxury cars, how many emirs are there at the party?

Since there are five luxury cars for each emir,	Number of luxury cars $=5\cdot$ Number of emirs
Since division is the inverse of multiplication,	Number of luxury cars $\div 5 =$ Number of emirs
Now the number of luxury cars is 35, therefore	$35 \div 5 =$ Number of emirs
Therefore	Number of emirs $= 7$

We would therefore answer the problem "7 emirs." Does it make sense to you that the answer is 35 ÷ 7?

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[35] An emir is a Muslim ruler. Why would emirs have so many luxury cars? Because Islam is a carnal, false religion which, along with other sects such as Protestantism and Talmudic Judaism, has waged war on the One, True Catholic Faith for centuries. Islam promises great worldly goods and pleasures to its followers. St. Thomas Aquinas, in his *Summa Contra Gentiles*, Book I, Ch. 6, echoes the consistent spirit of the saints, popes, Fathers, and Doctors before Vatican II, saying the following about Islam, its false promises, and its supposed "prophet" Muhammad: "*On the other hand, those who founded sects committed to erroneous doctrines proceeded in a way that is opposite to this. The point is clear in the case of Muhammad. He seduced the people by promises of carnal pleasure to which the concupiscence of the flesh goads us. His teaching also contained precepts that were in conformity with his promises, and he gave free rein to carnal pleasure. In all this, as is not unexpected, he was obeyed by carnal men. As for proofs of the truth of his doctrine, he brought forward only such as could be grasped by the natural ability of anyone with a very modest wisdom. Indeed, the truths that he taught he mingled with many fables and with doctrines of the greatest falsity. He did not bring forth any signs produced in a supernatural way, which alone fittingly gives witness to divine inspiration; for a visible action that can be only divine reveals an invisibly inspired teacher of truth. On the contrary, Muhammad said that he was sent in the power of his armaments [of war]—which are signs [which are] not lacking even to robbers and tyrants. What is more, no wise men, men trained in things divine and human, believed in him from the beginning. Those who believed in him were brutal men and desert wanderers, utterly ignorant of all divine teaching, through whose numbers Muhammad forced others to become his followers by the violence of his arms. Nor do divine pronouncements on the part of preceding prophets offer him any witness. On the contrary, he perverts almost all the testimonies of the Old and New Testaments by making them into fabrications of his own, as can be seen by anyone who examines his law. It was, therefore, a shrewd decision on his part to forbid his followers to read the Old and New Testaments, lest these books convict him of falsity. It is thus clear that those who place any faith in his words believe foolishly.*"

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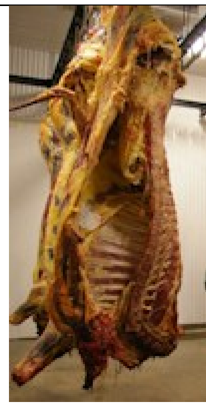
Now solve for x, *making sure to express the answer in the required units!*

$$x = \frac{(1600\ lbs.)(4.1)}{100}$$

$$x = 16\ lbs. \cdot 4.1 = \mathbf{65.6\ lbs.}$$

A new-born calf weighs 65.6 lbs.

Trickier percentage problems

		
Live weight = 100%	Carcass weight ("hanging weight") is about 60% of *live* weight	The final cuts of meat, all taken together, weigh about 70% of the *hanging* weight.
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You may suspect our next few problems below are far-fetched and unrealistic problems. But they are quite applicable to the real world. To prepare you for these problems, let us return to cows and consider this very common situation. When raising a beef cow on one's pasture, naturally one wonders how much of the cow's live weight will end up as meat in the freezer as roasts, steaks, and hamburger. The answer to this question is usually explained like this: The weight of the cow while still alive is called the "live weight." When the cow is slaughtered, the first thing they do is to drain the blood, discard the organs, head, hide, and other such things (unless the customer asks for them). The result of this butchering process is a carcass with just meat, bones, and fat, and the weight of the carcass is called the "hanging weight." The hanging weight is about 60% of the live weight. That is, we lose about 40% of the live weight during the butchering process.

But the butcher is not yet done. The final step is to cut the carcass up into small cuts of packaged meat for the home freezer. During this process, bones and fat are removed. In the end, the total weight of the packages of meat which end up in the customer's freezer is about 70% of the *hanging* weight.

Problem. A black angus beef cow weighs 1500 pounds while alive, grazing in the pasture. How many pounds of packaged meat will the customer actually receive from this cow?

The form of the answer should be something like, "The customer will get 500 pounds of meat", etc.

To solve this problem, we recall the key facts: 40% of the live weight is lost during the butchering process, yielding the "hanging weight." Then, during the final processing, only 70% of the hanging weight ends up as actual cuts of meat. Thus, we have two problems to work out to find the final weight.

Part 1: Determine the hanging weight

Write out the part/whole formula. Write x for the part since this is what we are solving for.

$$\frac{x}{whole} = \frac{percent}{100}$$

Plug in the values we are given for determination of the hanging weight. It is 60% of the live weight of the cow.

$$\frac{x}{1500\ lbs.} = \frac{\mathbf{60}}{100}$$

Now solve for x, *making sure to express the answer in the required units!*

$$x = \frac{(1500\ lbs.)(60)}{100}$$

$$x = \frac{90000\ lbs.}{100}$$

$$x = 900\ \boldsymbol{lbs.}$$

The hanging weight is 900 lbs.

Part 2: Determine the final, packaged meat weight

Write out the part/whole formula. Write x for the part since this is what we are solving for.

$$\frac{y}{whole} = \frac{percent}{100}$$

Plug in the values we are given for determination of the hanging weight. It is 70% of the hanging weight of the cow.

$$\frac{y}{900\ lbs.} = \frac{\mathbf{70}}{100}$$

Now solve for y, *making sure to express the answer in the required units!*

$$y = \frac{(900\ lbs.)(70)}{100}$$

$$y = \frac{63000\ lbs.}{100}$$

$$y = 630\ \boldsymbol{lbs.}$$

The customer gets 630 lbs. of packaged meat

HINT: If you are good with decimals, you might have noticed that you could solve the above problem *much* easier with this formula: customer pounds = (1500 live weight lbs.)(.6)(.7) = 630 lbs.

Although the above scenario concerned cows, there are countless other such scenarios in science, industry, economics, and so on.

Problem. A number is increased by **40**%, and then decreased by **25**%. The resulting number will be what percent of the original?

Notice here that the question is *not* asking for a particular number. Rather, the general answer form we are looking for is, "The resulting number will be 85% of the original number," whatever the correct percentage turns out to be. The problem could be restated like this:

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Lesson 42 - Solving: Mixture Problems with One Variable

One application of systems of equations are mixture problems. Mixture problems are ones in which two different solutions are mixed together, resulting in a new final solution. We will use the following table to help us solve mixture problems:

	Amount	Part	Total
Item 1			
Item 2			
Final			

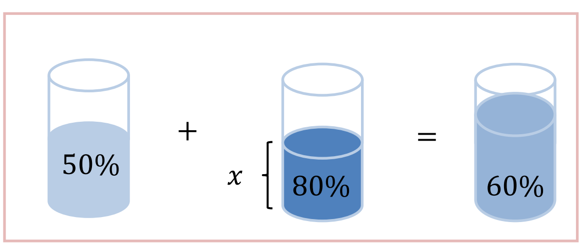
The first column is for the amount of each item we have. Into the second column, labeled *part*, we will put the rate of percentages we mix the rate (written as a decimal). (If the problem is about prices, we will put those prices in this column.) Then we can multiply the *amount* column by the *part* to find the *total* of each item by itself, that is, not diluted. Then we can derive an equation by adding the amount and/or total columns; this equation will help us solve the problem and answer the questions.

These kinds of problems can have either one or two variables. We will start with one-variable problems. To understand the first example problem below, know that men who work in laboratories mixing chemicals often have a need to prepare extremely precise mixtures consisting of the same kind, or different kinds of chemicals. One of types of preparations is called a "**solution**" – which is a chemical which has been dis**solved** in some other liquid. You could make a simple solution by dissolving a teaspoon of table salt in pure water, making a saltwater solution. If you wanted that twice as strong, you of course, would dissolve two teaspoons, and so on. You could do the same with sugar, or other, complex substances.

Problem. A chemist has 70 mL of a 50% methane solution. How much of an 80% solution must he add so that the final solution is 60% methane?

The image below illustrates the goal of the chemist and the nature of this problem. He already has a liquid solution which is composed of 50% methane. He wants to increase that concentration so that it will end up being 60% strong. **In this problem, we recognize the "how much" we are asked; we are seeking an answer in the form, "x number of milliliters should be added."**

Now, one way to make a weak solution stronger is to add some super-concentrated solution to it. Note that decimals are generally easier to work with rather than percentages. This is easy enough, for we just take a percentage such as 60%, remove the % sign and put a decimal in front. So 60% = .6



First, we set up the mixture table. The "Amount" we start with is 70 ml, but we don't know how many ml we should add; that is x. In the "Part" column, put the percentages written as decimals. Thus, 0.5 for start, 0.8 for add.

	Amount (in milliliters)	Part	Total
Start	70	0.5	
Add	x	0.8	
Final	$70 + x$	0.6	

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